On the Enumeration of Certain Weighted Graphs

Miklós Bóna* Hyeong-Kwan Ju† Ruriko Yoshida‡

February 2, 2008

Abstract

We enumerate weighted graphs with a certain upper bound condition. We also compute the generating function of the numbers of these graphs, and prove that it is a rational function. In particular, we show that if the given graph is a bipartite graph, then its generating function is of the form $\frac{p(x)}{(1-x)^{m+1}}$, where m is the number of vertices of the graph and $p(x)$ is a polynomial of degree at most m.

Key words Weighted Graphs, Rational Convex Polytopes, Rational Generating Functions, Ehrhart (Quasi-)polynomial.

1 Introduction

For a given nonnegative integer n, let $[n] := \{0, 1, 2, \cdots, n\}$. Also, let $G = (V, E)$ be a simple graph (no loops and no multiple edges allowed) with the vertex set $V = \{v_1, v_2, \cdots, v_m\}$. Let $\alpha = (n_1, n_2, \cdots, n_m) \in [n]^m$ so that

$$n_i + n_j \leq n \quad \text{if} \quad v_i v_j \in E. \tag{1}$$

In other words, the sum of two weights corresponding to an adjacent pair of vertices is bounded by a given integer n.

We call a triplet $WG_\alpha = (V, E, \alpha)$ a **weighted graph** of G with distribution α. We will denote by $WG(n)$ the number of all weighted graphs of G with a fixed upper bound n. Note that $WG(n)$ enumerates the set of all solutions of the system of linear inequalities corresponding to the graph G by the condition

*`bona@math.ufl.edu`, Department of Mathematics, University of Florida, Gainesville, FL 32611 USA.

†`hkju@chonnam.ac.kr`, Department of Mathematics, Chonnam National University, Kwangju, 500-757, Republic of Korea. The research of this author was financially supported by Chonnam National University.

‡`ruriko@math.duke.edu`, Department of Mathematics, Duke University, Durham, NC 27708-0320 USA.

(1).

Let $\mathcal{G}$ be the set of all simple graphs and $\mathbf{C}[[x]]$ the ring of formal power series. Define a map

$$\rho : \mathcal{G} \to \mathbb{C}[[x]] \ (G \ \mapsto \rho(G) = \sum_{n=0}^{\infty} WG(n)x^n).$$

For example, consider the following graph G:

$$\begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & b \arrow[r, dash] & c \cdot \\ d & e & \end{tikzcd}$$

For each $a = i \in [n]$, $\quad b, d, e$ can be $0, 1, 2, \cdots, n - i$, and for each $b = j \in [n - i]$, c can be $0, 1, 2, \cdots, n - j$. Hence,

$$\begin{aligned} WG(n) &= \sum_{i=0}^{n}(n+1-i)^2(\sum_{j=0}^{n-i}(n+1-j)) \\ &= \frac{(3n+5)(2n+3)(3n+4)(n+2)(n+1)}{120} \\ &= \frac{1}{120}(18(n+1)^5 + 45(n+1)^4 + 40(n+1)^3 + 15(n+1)^2 + 2(n+1)), \end{aligned}$$

and

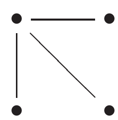
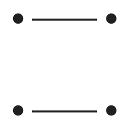
$$\begin{aligned} \rho(G) &= \frac{1}{120}(18\frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6} + 45\frac{1+11x+11x^2+x^3}{(1-x)^5} \\ &\quad + 40\frac{1+4x+x^2}{(1-x)^4} + 15\frac{1+x}{(1-x)^3} + 2\frac{1}{(1-x)^2}) \\ &= \frac{(1+x)(1+7x+x^2)}{(1-x)^6} = \frac{1+8x+8x^2+x^3}{(1-x)^6}. \end{aligned}$$

Equivalently, $WG(n)$ is the number of solutions $(n_a, n_b, n_c, n_d, n_e) \in [n]^5$ to the following system of inequalities:

$$\begin{cases} n_a + n_b \le n \\ n_b + n_c \le n \\ n_a + n_d \le n \\ n_a + n_e \le n \end{cases}$$

Remark 1.1.

1. *The map ρ is not injective.*

For graphs G_1  *and* G_2 

$$\rho(G_1) = \rho(G_2) = \frac{1 + 4x + x^2}{(1-x)^5} \quad \textit{since} \quad \sum_{k=1}^{n+1} k^3 = (\sum_{k=1}^{n+1} k)^2.$$

2. *We used the following identities (to get* $\rho(G)$ *in the previous example):*

$$(n+1)^m = \sum_{k=1}^{n} A(m,k)C(n+k,m)$$

where $A(m,k)$ *is an Eulerian number, and* $C(n,m) = \frac{n!}{m!(n-m)!}$ *is a binomial coefficient (See Fact 2.5 3. in Section 2.3.), and*

$$\sum_{n=0}^{\infty} C(n+k,k)x^n = \frac{1}{(1-x)^{k+1}}.$$

In Section 2 we compute $\rho(G)$ for linear graphs, circular graphs, complete graphs, star graphs, discrete graphs, a cubic graph, an octahedral graph and complete bipartite graphs. In Section 3 we remind a reader of rational convex polytopes and rational generating functions, and describe the relationship between these objects and our problem (enumeration of weighted graphs). In Section 4 we list many new problems that we encountered while we worked on this topic.

2 Generating Functions for Various Graphs

2.1 Linear Graphs

For the null graph $\emptyset$ we set $W\emptyset(n) = 1$ for all nonnegative integers n, so

$$\rho(\emptyset) = \frac{1}{1-x}.$$

Let L_1 be a one-vertex graph, and for $i \geq 2$ let L_i be the linear graph with i vertices. That is, L_i is a tree with two vertices of degree one and $i-2$ vertices of degree of two. For the one-vertex graph $L_1 = K_1(\bullet)$, $\quad WL_1(n) = n+1$ and

$$\rho(L_1) = \frac{1}{(1-x)^2}.$$

For $L_2 = K_2$ ($a\bullet$ ——— $\bullet b$) let $a = i,\ b = j$, and

$$b_{ij} = \begin{cases} 1 & \text{if } i+j \le n, \\ 0 & \text{otherwise} \end{cases} \quad = \quad \lfloor \frac{2n+3-i-j}{n+2} \rfloor.$$

Note that all indices i and j run from 0 to n, not from 1 to $n+1$. $B(n) := (b_{ij})$ is an $(n+1) \times (n+1)$ matrix, $J_n := (1, 1, 1, \cdots, 1)^t \in [n]^{n+1}$. Then

$$WL_2(n) = J_n^t B(n) J_n = \sum_{i,j=0}^{n} b_{ij} = C(n+2, 2),$$

$$\rho(L_2) = \frac{1}{(1-x)^3}.$$

For L_3 ($\bullet$ —— $\bullet$ —— $\bullet$),

$$WL_3(n) = J_n^t B(n)^2 J_n = C(n+3, 3) + C(n+2, 3),$$

$$\rho(L_3) = \frac{1+x}{(1-x)^4}.$$

By induction on k we can show that

$$WL_{k+1}(n) = J_n^t B(n)^k J_n$$

holds for all nonnegative integers k.

For example,

$$B(4) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B(4)^2 = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$B(4)^3 = \begin{bmatrix} 15 & 14 & 12 & 9 & 5 \\ 14 & 13 & 11 & 8 & 4 \\ 12 & 11 & 9 & 6 & 3 \\ 9 & 8 & 6 & 4 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}, \quad B(4)^4 = \begin{bmatrix} 55 & 50 & 41 & 29 & 15 \\ 50 & 46 & 38 & 27 & 14 \\ 41 & 38 & 32 & 23 & 12 \\ 29 & 27 & 23 & 17 & 9 \\ 15 & 14 & 12 & 9 & 5 \end{bmatrix}.$$

Note that $WL_{k+1}(n)$ is the $(0,0)$-entry of $B(n)^{k+2}$. For example, $WL_3(4) = 55$, which is the sum of all entries of the matrix $B(4)^2$.

Theorem 2.1 (Bona and Ju, 2006 [8]). *Let*

$$F(n,x) = \sum_{k=0}^{\infty} WL_{k+1}(n)x^k = \sum_{k=0}^{\infty} (J_n^t B(n)^k J_n)x^k.$$

Then

$$F(n,x) = \frac{1+F(n-1,-x)}{1-x(F(n-1,-x)+1)} = \cfrac{1}{-x+\cfrac{1}{1+F(n-1,-x)}}$$

$$= \cfrac{1}{-x+\cfrac{1}{1+\cfrac{1}{x+\cfrac{1}{1+F(n-2,x)}}}} := [-x,1,x,1,F(n-2,x)],$$

where $F(0,x) = \frac{1}{-x+1} = [-x,1]$ *and* $F(1,0) = \frac{2+x}{1-x-x^2} = [-x,1,x,1]$.

Theorem 2.2 (Bona and Ju, 2006 [8]).

$$F(n,x) = \frac{P_n(x)}{Q_n(x)},$$

where

$$Q_n(x) = \det(I - xB(n)) = \sum_{k=0}^{n} \binom{\lfloor \frac{n+k+1}{2} \rfloor}{k} (-1)^{\lfloor \frac{k+1}{2} \rfloor} x^k$$

and

$$P_n(x) = (Q_{n-2}(x) - (1+x)Q_n(x))/x^2$$

is a polynomial of degree $deg(Q_n(x)) - 1 = n$.

Remark 2.3.

1. $Q_n(x)$ *satisfies a recurrence relation*

$$Q_n(x) + (x^2-2)Q_{n-2}(x) + Q_{n-4}(x) = 0. \tag{2}$$

2. *From the recurrence relation* (2) *we get the following generating function* $R(x,y)$ *for the sequence* $\{Q_n(x)\}_{n=0}^{\infty}$*:*

$$R(x,y) = \sum_{n=0}^{\infty} Q_{n-1}(x)y^n = \frac{(1+y)(1-xy-y^2)}{1-y^2(2-x^2-y^2)}.$$

3. $\{F(n,x)\}_{n=0}^{\infty}$ *is an approximation(called Páde approximant) to the infinite periodic (of period 4) continued fraction*

$$[-x, 1, x, 1, -x, 1, x, 1, \cdots] = \frac{-(2+x) \pm \sqrt{x^2-4}}{2x},$$

and converges to $\begin{cases} \frac{-(2+x)+\sqrt{x^2-4}}{2x} & on \quad [2, \infty), \\ \frac{-(2+x)-\sqrt{x^2-4}}{2x} & on \quad (-\infty, 2]. \end{cases}$

4. *The explicit expression of the generating function* $G(x,y)$ *for the sequence* $\{F(n,x)\}_{n=0}^{\infty}$ *is not known, but from Theorem 2.2*

$$G(x,y) = \sum_{n=0}^{\infty} F(n,x)y^n = \sum_{n=0}^{\infty} \frac{Q_{n-2}(x)}{x^2 Q_n(x)} y^n - \frac{1+x}{x^2(1-y)}.$$

5. *Let* $s(M)$ *be the sum of all entries of the* $m \times m$ *matrix* M *with nonnegative integral entries, and let* $\bar{M}^{(k)}(x)$ *be the* $m \times m$ *matrix* $I - xM$ *where the* k*-th row of the matrix* $I - xM$ *is replaced by* $(1, 1, \cdots, 1)$. $f_k(x)$ *be the determinant of the matrix* $\bar{M}^{(k)}(x)$. *Then the generating function* $\eta(x)$ *for the sequence* $\{s(M^n)\}_{n=0}^{\infty}$ *is as follows (the proof is immediate from the 4.7.2 Theorem of R. Stanley [21]) :*

$$\eta(x) = \sum_{n=0}^{\infty} s(M^n)x^n = \frac{\sum_{i=1}^{m} f_i(x)}{\det(I - xM)}.$$

2.2 Circular Graphs

Before considering circular graphs, we first consider the linear graphs L_{k+1} with $k+1$ vertices, two of them being end vertices a and b, as follows below:

$a\bullet$ —— $\bullet$ —— $\bullet \cdots \bullet$ —— $\bullet$ —— $\bullet$ —— $\bullet b$

If we fix $n_a = i$ and $n_b = j$, then the number of all possible ways to give distributions corresponding to the vertex set of the graph L_{k+1} is same as the $(i,j)-$entry of the matrix $B(n)^k$.

Next, we identify the leftmost vertex a and the rightmost vertex b, resulting in the circular graph C_k with k vertices.

Hence, the number of all possible ways to give distributions corresponding to the vertex set of the graph C_k (with identified vertex having $n_a = n_b = i$ fixed) is same as the (i,i)-entry of the matrix $B(n)^k$.

This implies that $WC_k(n) = tr(B(n)^k)$ for $k \geq 2$, and $WC_1(n) = trace(I_{k+1}) = n+1$.

Lemma 2.4 (Theorem 4.7.2 and Corollary 4.7.3, Stanley, 1997 [21]). *For any $m \times m$ matrix M, we have:*

$$\sum_{l=1}^{\infty} tr(M^l)x^l = -\frac{x\frac{d}{dx}(\det(I - xM))}{\det(I - xM)}.$$

By the Lemma 2.4 above

$$CF(n,x) = \sum_{k=1}^{\infty} tr(B(n)^k)x^k = \lfloor \frac{n+1}{2} \rfloor x - \frac{xQ_n{}'(x)}{Q_n(x)}$$

and

$$CG(x,y) = \sum_{k=1,n=0}^{\infty} WC_k(n)x^k y^n = \frac{xy}{(1+y)(1-y)^2} - \sum_{n=0}^{\infty} \frac{xQ_n{}'(x)}{Q_n(x)} y^n.$$

We list several $\rho(C_i)$'s and $\rho(L_i)$'s for reference.

$\rho(C_1) = \frac{1}{(1-x)^2}$ $\rho(L_1) = \frac{1}{(1-x)^2}$
$\rho(C_2) = \frac{1}{(1-x)^3}$ $\rho(L_2) = \frac{1}{(1-x)^3}$
$\rho(C_3) = \frac{1+x+x^2}{(1+x)(1-x)^4}$ $\rho(L_3) = \frac{1+x}{(1-x)^4}$
$\rho(C_4) = \frac{1+2x+x^2}{(1-x)^5}$ $\rho(L_4) = \frac{1+3x+x^2}{(1-x)^5}$
$\rho(C_5) = \frac{1+6x+11x^2+6x^3+x^4}{(1+x)(1-x)^6}$ $\rho(L_5) = \frac{1+7x+7x^2+x^3}{(1-x)^6}$

For example, $WC_3(n) = \frac{1}{16}(4n^3 + 18n^2 + 28n + 15 + (-1)^n)$
$= \begin{cases} \frac{1}{8}(n+2)(2n^2+5n+4) & \text{if } n \text{ is even} \\ \frac{1}{8}(n+1)(2n^2+7n+7) & \text{if } n \text{ is odd} \end{cases}.$

2.3 Discrete Graphs

In order to describe discrete graphs, complete graphs, star graphs, complete bipartite graphs, etc, we need the notion of Eulerian numbers and Eulerian polynomials.

Let $p = p_1 p_2 \cdots p_t$ be a t-permutation. We say that i is a descent of p if $p_i > p_{i+1}$. Let $A(t,k)$ be the number of t-permutations with $k-1$ descents. The numbers $A(t,k)$ are called the **Eulerian numbers**, and $A_t(x) = \sum_{k=1}^{t} A(t,k)x^k$ is called the **Eulerian polynomial**.

Fact 2.5. *We list below several known facts about Eulerian numbers and Eulerian polynomials. [See Bóna, 2004 and 2005 [6, 7] for details about Eulerian numbers, and also Graham et. al., 1994 [15]]*

1. $A(t,k) = kA(t-1,k) + (t-k+1)A(t-1,k-1)$.
2. $A(t,k) = A(t,t+1-k)$ *and* $\sum_{k=1}^{t} A(t,k) = t!$.

3. $x^t = \sum_{k=1}^{t} A(t,k)C(x-1+k,t)$, *where* $C(t,k) = \frac{t!}{k!(t-k)!}$.

4. $g(x,y) = \sum_{t=0}^{\infty}\sum_{k=1}^{t} A(t,k)x^k \frac{y^t}{t!} = \sum_{t=0}^{\infty} A_t(x)\frac{y^t}{t!} = \frac{1-x}{1-xe^{y(1-x)}}$.

5. $\frac{A_t(x)}{x(1-x)^{t+1}} = \frac{d}{dx}\{\frac{A_{t-1}(x)}{(1-x)^t}\}$.

6. $A_t(x) = txA_{t-1}(x) + x(1-x)A'_{t-1}(x)$.

7. $A_t(x) = x(1-x)^t + \sum_{k=1}^{t} C(t,k)(1-x)^{t-k}A_k(x)$.

8. $A(t,k) = \sum_{i=0}^{k}(-1)^i C(t+1,i)(k-i)^n$.

Let $D_t = (V,E)$ be a discrete graph, that is, a graph with $|V| = t$ and $E = \emptyset$. Then, clearly,

$$WD_t(n) = (n+1)^t, \quad \rho(D_t) = \frac{A_t(x)}{(1-x)^{t+1}},$$

and

$$F(x,y) = \sum_{t=0}^{\infty} \rho(D_t)\frac{y^t}{t!} = \frac{1}{1-xe^y}.$$

2.4 Complete Graphs

Let K_t be a complete graph of order t. Then,

Theorem 2.6 (Ju, 2006 [16]).

$$WK_t(n) = t\sum_{r=1}^{\lfloor\frac{n+1}{2}\rfloor} r^{t-1} + (\lfloor\frac{n+2}{2}\rfloor)^t,$$

and

$$F(x,y) := \sum_{t=0}^{\infty}\sum_{n=0}^{\infty} WK_t(n)x^n\frac{y^t}{t!} = (1+x+\frac{xy}{1-x})\frac{e^y}{1-x^2e^y}.$$

We define

$$\alpha(y;x) = 1+x+\frac{xy}{1-x}, \qquad \beta(y;x) = \frac{e^y}{1-x^2e^y}.$$

Using the facts given in Section 2.3 we can show that the following holds:

$$\begin{aligned}\beta(y;x) = \frac{e^y}{1-x^2e^y} &= (\sum_{t=0}^{\infty}\frac{y^t}{t!})(\frac{1}{1-x^2} + \sum_{t=1}^{\infty}\frac{A_t(x^2)}{(1-x^2)^{t+1}}\frac{y^t}{t!}) \\ &= \frac{1}{1-x^2} + \sum_{t=1}^{\infty}\frac{A_t(x^2)}{x^2(1-x^2)^{t+1}}\frac{y^t}{t!},\end{aligned}$$

and

$$F(x,y) = \frac{1}{1-x} + \frac{y}{(1-x)^2} + \sum_{t=2}^{\infty} \frac{A_t(x^2) + txA_{t-1}(x^2)}{x^2(1-x)(1-x^2)^t} \frac{y^t}{t!}. \tag{3}$$

For $t \geq 2$, the numerator of the summand in the equation (3) is

$$\frac{A_t(x^2) + txA_{t-1}(x^2)}{x^2} = (1+x)\frac{txA_{t-1}(x^2) + x^2(1-x^2)A'_{t-1}(x^2)}{x^2}.$$

Let

$$r_t(x) = \frac{txA_{t-1}(x^2) + x^2(1-x^2)A'_{t-1}(x^2)}{x^2},$$

for $t = 2, 3, \cdots$. By Fact 2.5 2. (symmetric condition),

$$r_t(-1) = \sum_{k=1}^{t-1}(2k-t)A(t-1,k) = 0,$$

for $t = 2, 3, \cdots$. This implies that

$$\frac{A_t(x^2) + txA_{t-1}(x^2)}{x^2} = (1+x)^2 p_{2t-4}(x)$$

for some polynomial $p_{2t-4}(x)$ in x of degree $2t-4$.

Theorem 2.7. *For $U_t(x) = \sum_{n=0}^{\infty} WK_t(n)x^n$,*

$$F(x,y) = \sum_{t=0}^{\infty} U_t(x)\frac{y^t}{t!} = \frac{1}{1-x} + \frac{y}{(1-x)^2} + \sum_{t=2}^{\infty} \frac{p_{2t-4}(x)}{(1-x)^3(1-x^2)^{t-2}} \frac{y^t}{t!},$$

where

$$p_{2t-4}(x) = \frac{A_t(x^2) + txA_{t-1}(x^2)}{x^2(1+x)^2}$$

is a polynomial of degree $2t-4$ for $t = 2, 3, 4, \cdots$.

Computations (using Maple) of $U_t(x)$ for $t = 0, 1, 2, \cdots, 8$ have shown that:

$$
\begin{aligned}
U_0(x) &= \frac{1}{1-x}, \\
U_1(x) &= \frac{1}{(1-x)^2}, \\
U_2(x) &= \frac{1}{(1-x)^3}, \\
U_3(x) &= \frac{1+x+x^2}{(1-x)^3(1-x^2)}, \\
U_4(x) &= \frac{1+2x+6x^2+2x^3+x^4}{(1-x)^3(1-x^2)^2} \\
U_5(x) &= \frac{1+3x+19x^2+14x^3+19x^4+3x^5+x^6}{(1-x)^3(1-x^2)^3} \\
U_6(x) &= \frac{1+4x+48x^2+56x^3+142x^4+56x^5+48x^6+4x^7+x^8}{(1-x)^3(1-x^2)^4} \\
U_7(x) &= \frac{1+5x+109x^2+176x^3+730x^4+478x^5+730x^6}{(1-x)^3(1-x^2)^5} \\
&\quad + \frac{176x^7+109x^8+5x^9+x^{10}}{(1-x)^3(1-x^2)^5}
\end{aligned}
$$

Remark 2.8.

1. *The sequence $a_1, a_2, \cdots, a_n$ of positive real numbers is called unimodal if there exists an index k such that $1 \leq k \leq n$, and $a_1 \leq a_2 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$. The same sequence is called log-concave if $a_{k-1}a_{k+1} \leq a_k^2$ holds for all indices k. It is well-known that a log-concave sequence is unimodal.(See Bona[7] for the proof.) The sequence $\{A(t,k\}_{k=1}^t$ of Eulerian numbers is log-concave, so it is unimodal for all t. The coefficients of numerator in $U_t(x)$ is unimodal for $t = 1, 2, 3, 4, 6$, but not always as shown in $U_5(x), U_7(x)$ above.*

2. *The rational functions $U_t(x)$ have denominator $(1-x^2)$. This means that $WK_t(n)$ is a (in fact, Ehrhart)* **quasi-polynomial** *of a certain polytope.(We will mention this later in Section 3.) Hence, its form depends on the parity of n. (See the next remark.)*

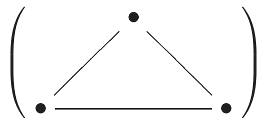
3. *The sequence* $\{WK_t(n)\}_{t,n}$*(for* $t = 0, 1, 2, \cdots, 5$*) are provided below.*

$WK_0(n) = C(n, 0) = 1$ *(empty graph)*

$WK_1(n) = C(n+1, 1) = n + 1$ (•)

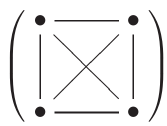
$WK_2(n) = C(n+2, 2) = \frac{1}{2}(n^2 + 3n + 2)$ (• —— •)

$WK_3(n) = \frac{1}{16}(4n^3 + 18n^2 + 28n + 15 + (-1)^n)$



(Remarks 4 of Section 6 of Bona and Ju[8])

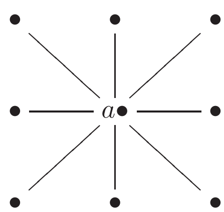
$WK_4(n) = \frac{1}{16}(2n^4 + 12n^3 + 28n^2 + 30n + 13$
$+ (-1)^n(2n + 3))$



$WK_5(n) = \frac{1}{192}(12n^5 + 90n^4 + 280n^3 + 450n^2 + 374n + 129$
$+ (-1)^n(30n^2 + 90n + 63))$

2.5 Star Graphs

The *Star Graph* S_t of order t is a tree with $t+1$ vertices, one of them of degree t and all others of degree 1.



: S_8

If we let the hub vertex a have value i, then the rest of all the vertices must have values in $[n-i]$. So

$$WS_t(n) = \sum_{i=0}^{n}(n+1-i)^t = \sum_{k=1}^{n+1} k^t = \sum_{k=1}^{n+1}[\sum_{i=1}^{t} A(t,i)C(k-1+i,t)]$$
$$= \sum_{i=1}^{t} A(t,i)C(n+1+i,t+1)$$

by Fact 2.5 3. in Section 2.3.

Now

$$\rho(S_t) = \sum_{n=0}^{\infty} WS_t(n)x^n = \sum_{i=1}^{t} \frac{x^{t-i}}{(1-x)^{t+2}} A(t,i) = \frac{A_t(x)}{x(1-x)^{t+2}},$$

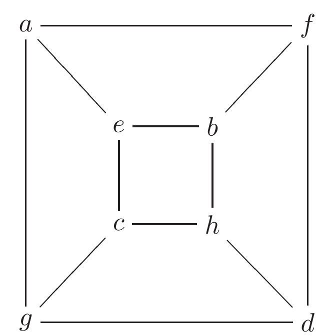
and

$$F(x,y) = \sum_{t=0}^{\infty} \rho(S_t)\frac{y^t}{t!} = \sum_{t=0}^{\infty} \frac{A_t(x)}{x(1-x)^{t+2}}\ \frac{y^t}{t!} = \frac{1}{x(1-x)(1-xe^y)},$$

by Fact 2.5 4.

2.6 Cubic Graphs

The *Cubic Graph* C is defined as follows:



Given four numbers $a, b, c, d \in [n], e \in [n - max(a,b,c)]$, $f \in [n - max(a,b,d)], g \in [n - max(a,c,d)]$, and $h \in [n - max(b,c,d)]$.

Hence

$$\begin{aligned}
WC(n) = &\sum_{a,b,c,d=0}^{n} (n+1-max(a,b,c))(n+1-max(a,b,d)) \\
&(n+1-max(a,c,d))(n+1-max(b,c,d)) \\
&= \sum_{d=0}^{n} \sum_{max(a,b,c)=0}^{n} (n+1-max(a,b,c))(n+1-max(a,b,d)) \\
&(n+1-max(a,c,d))(n+1-max(b,c,d)) \\
&= \sum_{d=0}^{n}\Big(\sum_{k=0}^{d} + \sum_{k=d+1}^{n}\Big) \sum_{max(a,b,c)=k} (n+1-max(a,b,c)) \\
&(n+1-max(a,b,d))(n+1-max(a,c,d))(n+1-max(b,c,d)) \\
&= \sum_{d=0}^{n}\sum_{k=0}^{d} \sum_{max(a,b,c)=k} (n+1-d)^3(n+1-k) \\
&+ \sum_{d=0}^{n-1}\sum_{k=d+1}^{n} \sum_{max(a,b,c)=k} (n+1-k) \\
&(n+1-max(a,b))(n+1-max(a,c))(n+1-max(b,c)) \\
&:= [I] + [II].
\end{aligned}$$

$$[I] = \frac{1}{3360}(n+1)(n+2)(n+3)(15n^5+138n^4+533n^3+1074n^2+1180n+560).$$

By the Inclusion Exclusion Principle,

$$\sum_{max(a,b,c)=k} (n+1-k)(n+1-max(a,b))(n+1-max(a,c))(n+1-max(b,c))$$

$$= 3\sum_{l=0}^{k}(n+1-k)^3(n+1-l)((l+1)^2-l^2) - 3\sum_{l=0}^{k}(n+1-k)^4 + (n+1-k)^4.$$

$$[II] = \frac{1}{3360}n(n+1)(n+2)(45n^5+357n^4+1177n^3+1971n^2+1638n+412).$$

$$\begin{aligned} WC(n) &= [I]+[II] \\ &= C(n+8,8)+26C(n+7,8)+175C(n+6,8)+316C(n+5,8) \\ &+ 175C(n+4,8)+26C(n+3,8)+C(n+2,8). \end{aligned}$$

$$\rho(C) = \frac{1+26x+175x^2+316x^3+175x^4+26x^5+x^6}{(1-x)^9}.$$

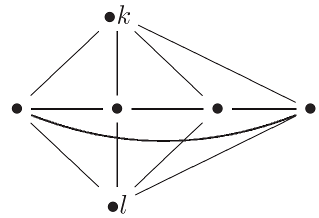
Let HC_d be a d-dimensional Hypercubic Graph. Then

$$\begin{aligned} \rho(HC_0) &= \frac{1}{(1-x)^2} \\ \rho(HC_1) &= \frac{1}{(1-x)^3} \\ \rho(HC_2) &= \frac{1+2x+x^2}{(1-x)^5} \\ \rho(HC_3) &= \frac{1+26x+175x^2+316x^3+175x^4+26x^5+x^6}{(1-x)^9} \\ \rho(HC_d) &= \frac{P_{2^d-2}(x)}{(1-x)^{1+2^d}}(d \geq 1) \end{aligned}$$

(We conjecture the last one(general case)!) where $P_{2^d-2}(x)$ is a symmetric polynomial of degree 2^d-2.

2.7 Octahedral Graphs

In this section we focus on octahedral graphs. An octahedral graph is the Platonic graph with six nodes and 12 edges having the connectivity of the octahedron. Let us consider the following graph.



Let OH be an octahedral graph given in the figure above. Let a top vertex have value k and a bottom vertex l as in the figure. Then rest of them have values in $[n-m]$, where $m = max(k,l)$. Let $B(n,m) = (b(n,m)_{ij})$ be an $(n+1)\times(n+1)$ matrix for which $b(n,m)_{ij} = 0$ if $max(i+j, max(i,j)+m) > n$ and $b(n,m)_{ij} = 1$ otherwise. For example,

$$B(4,0) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, B(4,1) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B(4,2) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B(4,3) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B(4,4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} WOH(n) &= \sum_{m=0}^{n} ((m+1)^2 - m^2) trace(B(n,m)^4) \\ WOH(2k) &= \sum_{m=0}^{k-1} (2m+1) trace(B(2k,m)^4) + \sum_{m=k}^{2k} (2m+1)(2k+1-m)^4 \\ &=: (I) + (II) \\ WOH(2k+1) &= \sum_{m=0}^{k} (2m+1) trace(B(2k+1,m)^4) + \sum_{m=k+1}^{2k+1} (2m+1)(2k+2-m)^4 \\ &=: (III) + (IV). \end{aligned}$$

In order to compute the quantities (I) and (III) above, we need to define the following. Let $D(n)$ be an $(n+1)\times(n+1)$ matrix all of whose entries are 1, and let $S(n,m) := (s_{ij})$ be an $(n+1)\times(n+1)$ matrix such that $s_{ij} = 1$ if $i+j > 2n-m$ and 0 otherwise.

For example,

$$D(4) - S(4,2) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

$$
\begin{aligned}
(D(r)-S(r,m))^4 &= (D(r)^2 - D(r)S(r,m) - S(r,m)D(r,m) + S(r,m)^2)^2 \\
&=: (M - DS - SD + K)^2 \\
&= M^2 - MDS - MSD + MK - DSM + DSDS + DSSD - DSK \\
&\quad - SDM + SDDS + SDSD - SDK + KM - KDS - KSD + K^2.
\end{aligned}
$$

Let us define

$$
\begin{aligned}
p(r,m) &:= trace((D(r-1) - S(r-1,m))^4) \\
&= r^4 - 4C(m+1,2)r^2 + 4[C(m+1,3) + C(m+2,3)]r \\
&\quad - 4C(m+2,4) - C(m+1,2).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
WOH(2k) &= \sum_{m=0}^{k-1} (2m+1)p(2k+1-m, 2(k-m)) \\
&\quad + \sum_{m=k}^{2k} (2m+1)(2k+1-m)^4 \\
&= \frac{1}{10}(k+1)(2k^2+2k+1)(12k^3+30k^2+27k+10), \\
WOH(2k+1) &= \sum_{m=0}^{k} (2m+1)p(2k+2-m, 2k+1-2m) \\
&\quad + \sum_{m=k+1}^{2k+1} (2m+1)(2k+2-m)^4 \\
&= \frac{1}{10}(k+1)(2k^2+6k+5)(12k^3+42k^2+51k+20).
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
WOH(n) &= \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} (2m+1)p(n+1-m, n-2m) \\
&\quad + \sum_{m=\lfloor \frac{n+1}{2} \rfloor}^{n} (2m+1)(n+1-m)^4 \\
&= \frac{1}{160}(6n^6 + 54n^5 + 210n^4 + 450n^3 + 559n^2 + 381n + 115 \\
&\quad + (-1)^n(10n^3 + 45n^2 + 75n + 45))
\end{aligned}
$$

and

$$
\rho(OH) = \frac{1 + 7x + 48x^2 + 89x^3 + 142x^4 + 89x^5 + 48x^6 + 7x^7 + x^8}{(1+x)^4(1-x)^7}
$$

2.8 Complete Bipartite Graphs

Let $K_{p,q} = (V, E)$ be a complete bipartite graph of order (p, q). So, $V = X \cup Y, E = \{ab = ba | a \in X, b \in Y\}, |X| = p, |Y| = q$. We also let $WK_{p,q,\alpha}$ be a weighted graph with a distribution α of the form $\alpha = ((n_1, n_2, \cdots, n_p), (m_1, m_2, \cdots, m_q))$ and $r = max(n_1, n_2, \cdots, n_p)$. Then $m_j \in [n - r](j = 1, 2, \cdots, q)$.

$$
\begin{aligned}
WK_{p,q}(n) &= \sum_{n_1,n_2,\cdots,n_p=0}^{n} (n + 1 - max(n_1, n_2, \cdots, n_p))^q \\
&= \sum_{k=0}^{n} ((k + 1)^p - k^p)(n + 1 - k)^q
\end{aligned}
$$

If we let $\bar{A}(t, k) = A(t, k + 1)$ and $\bar{A}_t(x) = \sum_{k=0}^{t-1} \bar{A}(t, k)x^k$. Then

$$
\rho(K_{p,q}) = \sum_{n=0}^{\infty} |WK_{p,q}(n)|x^n = \frac{\bar{A}_p(x)\bar{A}_q(x)}{(1 - x)^{p+q+1}}
$$

For example,

$$
\begin{aligned}
\bar{A}_3(x) &= \bar{A}(3, 0) + \bar{A}(3, 1)x + \bar{A}(3, 2)x^2 \\
&= 1 + 4x + x^2, \\
\bar{A}_4(x) &= \bar{A}(4, 0) + \bar{A}(4, 1)x + \bar{A}(4, 2)x^2 + \bar{A}(4, 3)x^3 \\
&= 1 + 11x + 11x^2 + x^3,
\end{aligned}
$$

and

$$
\rho(K_{3,4}) = \rho(K_{4,3}) = \frac{(1 + 4x + x^2)(1 + 11x + 11x^2 + x^3)}{(1 - x)^8}.
$$

Remark 2.9.

1. *We have* $H(x, y, z) := \sum_{p,q=0}^{\infty} \rho(K_{p,q}) \frac{y^p}{p!} \frac{z^q}{q!} = \frac{1-x}{(1-xe^y)(1-xe^z)}$

2. *We have* $\rho(K_{p,p}) = \frac{\bar{A}_p(x)^2}{(1-x)^{2p+1}}$.

3. *The graph* $K_{p,q}$ *has no cycles of odd length. Hence generating functions are not quasi-polynomials (as we will show later).*

4. *The degree of the numerator in* $\rho(K_{p,q})$ *is* $p + q - 2$ *and the numerator is symmetric.*

3 Rational Polytopes and Rational Generating Functions

Readers unfamiliar with this topic may wish to consult the book of R. Stanley([21], Chapter 4). A **quasi (or pseudo)-polynomial** of degree d with a quasi-period

N is a function $f : \mathbb{N} \to \mathbb{C}$ of the form $f(n) = \sum_{i=0}^{d} c_i(n)n^i$ where the coefficients are periodic functions of a common period N and the leading coefficient $c_d(n)$ is not identically zero.

Example 3.1. *The number $f(n)$ of unit squares in the region bounded by $x = 0, x = n(\geq 1), y = 0, y = \frac{4}{3}x$:*

$$f(n) = \begin{cases} \frac{4}{3}n - 2 & \textit{if } n \equiv 0(mod3) \\ \frac{4}{3}n - \frac{4}{3} & \textit{if } n \equiv 1(mod3) \\ \frac{4}{3}n - \frac{5}{3} & \textit{if } n \equiv 2(mod3) \end{cases}.$$

$c_1(n) = \frac{4}{3}$ *is constant(1-periodic), but $c_0(n)$ is 3-periodic.*

The generating function associated with the sequence $\{f(n)\}_{n=0}^{\infty}$ is as follows:

$$g(x) = \sum_{n=0}^{\infty} f(n)x^n = \frac{x^2(1 + x + 2x^2)}{(1 - x)(1 - x^3)}.$$

Here is another nontrivial example from the page 220 (Exercises 13.9 and 13.10) of Pach and Agarwal [18].

Example 3.2. *Given a set P of n points in the plane, for any $p \in P$, let $\mu_P(p)$ denote the number of farthest neighbors of p, that is,*

$$\mu_P(p) = |\{q \in P||p - q| = max_{r \in P}|p - r|\}|.$$

Let $\mu(n) = max_{|P|=n} \sum_{p \in P} \mu_P(p)$. It turned out (see Csizmadia [10] or Avis et al [2]) that if n is sufficiently large, then

$$\mu(n) = \frac{n^2}{4} + \frac{3n}{2} + \begin{cases} 3 & \textit{if } n \equiv 0(mod2), \\ \frac{9}{4} & \textit{if } n \equiv 1(mod4), \\ \frac{13}{4} & \textit{if } n \equiv 3(mod4). \end{cases}$$

The generating function associated with the sequence $\{\mu(n)\}_{n=0}^{\infty}$ is as follows:

$$g(x) = \sum_{n=0}^{\infty} \mu(n)x^n = \frac{3 - 3x + 4x^2 - x^3 - 3x^4 + 2x^5}{(1 - x)^2(1 - x^4)}.$$

If denominator in the reduced generating function associated with a certain sequence $\{f(n)\}_{n=0}^{\infty}$ has a factor $1 - x^N$ then $f(n)$ has a quasi-period N as in Example 3.1 $(1 - x^3)$ and Example 3.2 $(1 - x^4)$ [21].

Suppose we have a finite set of points $\{t_1, t_2, \cdots, t_d\}$ in $\mathbb{R}^m$. The **convex hull** of the set $\{t_1, t_2, \cdots, t_d\}$ is the set of all convex combinations of the given points, i.e. $\{x \in \mathbb{R}^m : x = \sum_{i=1}^{d} \lambda_i t_i, \lambda \geq 0, \sum_{i=1}^{d} \lambda_i = 1\}$. By a **convex polytope**, or simply a **polytope**, we mean a set which is the convex hull of a non-empty finite set $\{t_1, t_2, \cdots, t_d\}$. An **affine combination** of points $t_1, t_2, \cdots, t_k$ from $\mathbb{R}^m$ is a linear combination $\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_k t_k$, where

$\lambda_1 + \cdots + \lambda_k = 1$ and $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, k$. A k-family $(t_1, t_2, \cdots, t_k)$ of points from $\mathbb{R}^m$ is said to be **affinely independent** if a linear combination $\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_k t_k$ with $\lambda_1 + \cdots + \lambda_k = 0$ can only have the value 0 when $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0$. A polytope $\mathcal{P}$ with the property that there exists an affinely independent family $(t_1, t_2, \cdots, t_d)$ such that $\mathcal{P}$ is a convex hull of $\{t_1, t_2, \cdots, t_d\}$ is called a **simplex**. (Refer Barvinok [3], Brøndsted [9], Miller and Sturmfels [17], Stanley [20] or Schrijver [19] for details on polytopes.)

For a graph $G = (V, E) \in \mathcal{G}$, let $m = |V|$ and

$$\mathcal{P}(G) = \{(x_1, x_2, \cdots, x_m) \in \mathbb{R}^m_+ | x_i + x_j \leq 1 \text{ for all edge } v_i v_j \in E\}.$$

Note that for every simple graph G $\mathcal{P}(G)$ is a polytope which is contained in the m-dimensional unit hypercube. If all of the coordinates of the vertices of the polytope are integers, then we call such a polytope an **integer polytope** (or an **integral polytope**, a **lattice polytope**). If all of the coordinates of vertices of the polytope are rational numbers, then the associated polytope is called a **rational polytope**.

Note that every $\mathcal{P}$ is homeomorphic to a ball $\mathbf{B}^d$, for some d. This d is the dimension $dim(\mathcal{P})$ of the polytope $\mathcal{P}$. We denote the **boundary** (resp. **interior**) of $\mathcal{P}$ by $\partial\mathcal{P}$(resp. $\bar{\mathcal{P}}$). $\alpha \in \mathcal{P}$ is a **vertex** of $\mathcal{P}$ if there exists a closed affine half-space $\mathcal{H}$ such that $\mathcal{P} \cap \mathcal{H} = \{\alpha\}$

If $\mathcal{P} \in \mathbb{R}^m$ is a rational convex polytope and if $n \in \mathbb{Z}_+$, then we define $i(\mathcal{P}, n) := |n\mathcal{P} \cap \mathbb{Z}^m|$ and $\bar{i}(\mathcal{P}, n) := |n\bar{\mathcal{P}} \cap \mathbb{Z}^m|$.
This is called the **Ehrhart quasi-polynomial** of $\mathcal{P}$ and $\bar{\mathcal{P}}$ respectively. (We will show later that this is a quasi-polynomial.) If $\mathcal{P}$ is an integer polytope then this is simply called the **Ehrhart polynomial** of $\mathcal{P}$ and $\bar{\mathcal{P}}$ respectively.

If $\beta \subset \mathbb{Q}^m$, then define $den\beta$ (the denominator of β) as the least integer $q \in \mathbb{Z}_+$ such that $q\beta \subset \mathbb{Z}^m$.

Theorem 3.3 (Stanley, 1997 [21], pp237-238). *If $\mathcal{P}$ is a rational convex polytope of dimension d in $\mathbb{R}^m$, and*

$$F(\mathcal{P}, x) := 1 + \sum_{n=1}^{\infty} i(\mathcal{P}, n) x^n.$$

Then $F(\mathcal{P}, x)$ is a rational function $\frac{P(x)}{Q(x)}$ of x, where $deg(P(x)) \leq d$ and $Q(x)$ **can be** *written as $\prod_{\beta \in V}(1 - x^{den\beta})$. If $F(\mathcal{P}, x)$ is written in lowest terms, then $x = 1$ is a pole of order $d+1$, and no value of x is a pole of order $> d+1$. We also have the following (reciprocity for Ehrhart quasi-polynomial):*

$$\bar{i}(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n).$$

Theorem 3.3 says that $i(\mathcal{P}, n)$ is a **quasi-polynomial** with the correct value $i(\mathcal{P}, 0) = 1$, and $D(x) = \prod_{\beta \in V}(1 - x^{den\beta})$ is not in general the least denominator

of $F(\mathcal{P}, x)$. However, the least denominator has a factor $(1-x)^{d+1}$ but not $(1-x)^{d+2}$, while $D(x)$ has a factor $(1-x)^{|V|}$. See the following example (Stanley [21]).

Example 3.4. *Let $\mathcal{P}$ be the convex hull of the vertices set $V^* = \{(0,0,0), (1,0,0), (0,1,0), (1,1,0), (1/2, 0, 1/2)\}$. Then*

$$D(x) = \prod_{\beta \in V^*} (1 - x^{den \beta}) = (1-x)^5 (1+x),$$

but $F(\mathcal{P}, x) = \frac{1}{(1-x)^4}$

In order to prove our main theorem we need some information about the denominator of the vertex set for the polytope $\mathcal{P}(G)$.

Lemma 3.5. *Let $G = (V, E)$ with $|V| = m$ be a simple bipartite graph and V^* the set of vertices of the polytope $\mathcal{P}(G)$. Then $den(V^*) = \{1\}$. Otherwise (equivalently, if the simple graph G is not bi-colorable, or if it has a cycle of odd length), then $den(V^*) = \{1, 2\}$.*

Proof. If $G = (V, E)$ is bipartite. Then the defining matrix for the polytope $\mathcal{P}(G)$ is totally unimodular (Example 1 on p. 273, [19]). Thus, $den(V^*) = \{1\}$.

If $G = (V, E)$ is not bipartite. then, G contains at least a cycle with odd length. Consider a cycle with odd length as a subgraph and a submatrix in the defining matrix for the polytope $\mathcal{P}(G)$.

Let A be the defining matrix for G, and suppose the polytope $\mathcal{P}(G)$ is defined by the system $\{x \in \mathbb{R}^m : Ax \leq b,\ x \geq 0\}$ where A is a $s \times m$ matrix. Then we can rewrite this system as $\{x \in \mathbb{R}^m : A'x \leq b'\}$ where $A' = \begin{pmatrix} A \\ -I_m \end{pmatrix}$ where I_m is an identity matrix of order m and $b' = \begin{pmatrix} b \\ 0 \end{pmatrix}$ where 0 is a m dimensional vector with all zeros.

Then note that any vertex of the polytope $\mathcal{P}(G)$ is determined by the unique solution of a subsystem

$$A''x = b''$$

where A'' is a $m \times m$ minor of A' with $\det(A'') \neq 0$ and b'' is a m sub-vector of b' (Theorem 8.4 [19]). Thus all we have to show is that $\det(A'') = \{\pm 1, \pm 2\}$ if $\det(A'') \neq 0$. Then, there are three cases we have to consider.

1. All rows of A'' are from I_m.
2. All rows of A'' are from A.
   (a) A'' contains a cycle (or cycles) with odd length.
   (b) otherwise.
3. k rows of A'' are from A and $m-k$ rows are from I_m.

(a) A'' contains a cycle (or cycles) with smaller odd length k.

(b) otherwise.

If $A'' = I_m$, then clearly this defines the origin. For Case 2b and Case 3b, A'' does not contain a cycle with odd length. Thus we are done. For Case 2a and Case 3a, we have to show that the matrix for each subsystem of the system $A''x = b''$ defining a cycle with its length k, where k is an odd positive integer with $k \geq 3$, has its determinant ± 2. Let $\bar{A}$ be the defining matrix for a cycle with odd length. Note that $\bar{A}$ is a $k \times k$ matrix and note that after permuting columns and rows, $\bar{A}$ forms such that:

$$\bar{A} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Note that $\bar{A}$ is an upper triangular matrix except for the 1 in the first position of the kth row. So the determinant of $\bar{A}$ is clearly 2. Thus, $den(V^*) = \{1, 2\}$. $\square$

The next theorem is the conclusion of all that we discussed. Its proof follows immediately from Theorem 3.3 and Lemma 3.5.

Theorem 3.6. *If a simple graph $G = (V, E)$ with $|V| = m$ is bipartite (equivalently, $\chi(G) \leq 2$), then $WG(n) = i(\mathcal{P}(G), n)$ is an Ehrhart polynomial and*

$$\rho(G) = \frac{P(x)}{(1 - x)^{m+1}},$$

where $P(x)$ is a symmetric polynomial of degree $\leq m$. That is, if $WG(n)$ is an Ehrhart quasi-polynomial in n of a quasi-period $\neq 1$, then the quasi-period is 2, G contains a cycle of odd length (hence $\chi(G) > 2$) and

$$\rho(G) = \frac{P(x)}{(1 - x^2)^k (1 - x)^{m+1-k}},$$

where $P(x)$ is a polynomial of degree $\leq m + k$ and k is a nonnegative integer.

Corollary 3.7. *If a simple graph $G = (V, E)$ with $|V| = m$ is either a tree, a circular graph of even length, a discrete graph, a hypercubic graph, a complete bipartite graph, or a grid graph, then $\rho(G) = \frac{p(x)}{(1-x)^{m+1}}$, where $p(x)$ is a symmetric monic polynomial of degree at most m.*

Proof. All of those graphs are bipartite graphs. Hence, by Theorem 3.6, the result follows. $\square$

Example 3.8. *Consider the Circular Graph C_3 (a cycle of odd length 3).*

$\mathcal{P}(C_3) = \{(r,s,t) \in \mathbb{R}^3 | r,s,t \geq 0, r+s, s+t, t+r \leq 1\}$

$V^* = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(\frac{1}{2},\frac{1}{2},\frac{1}{2})\}$

$m = dim(\mathcal{P}(C_3)) = 3$

$WC_3(n) = i(\mathcal{P}(C_3), n) = \frac{1}{16}(4n^3 + 18n^2 + 28n + 15 + (-1)^n)$

$\rho(C_3) = F(\mathcal{P}(C_3), x) = 1 + \sum_{n=1}^{\infty} i(\mathcal{P}(C_3), n)x^n = \frac{1+x+x^2}{(1-x^2)(1-x)^3} = \frac{P(x)}{(1-x^2)^1(1-x)^{3+1-1}}$

$D(x) = \prod_{\beta \in V^*}(1 - x^{den\beta}) = (1-x^2)(1-x)^4.$

Example 3.9. *Consider the Circular Graph C_4 (a cycle of even length 4).*

$\mathcal{P}(C_4) = \{(r,s,t,u) \in \mathbb{R}^4 | r,s,t,u \geq 0, r+s, s+t, t+u, u+r \leq 1\}$,

$V^* = \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(0,1,0,1),(1,0,1,0)\}$,

$m = dim(\mathcal{P}(C_4)) = 4$,

$WC_4(n) = C(n+2,4) + 2C(n+3,4) + c(n+4,4) = 1 + \frac{5}{2}n + \frac{7}{3}n^2 + n^3 + \frac{1}{6}n^4$,

$\rho(C_4) = 1 + \sum_{n=1}^{\infty} i(\mathcal{P}(C_4), n)x^n = \frac{1+2x+x^2}{(1-x)^5} = \frac{P(x)}{(1-x)^{4+1}}$,

$D(x) = \prod_{\beta \in V^*}(1 - x^{den\beta}) = (1-x)^7.$

Example 3.10. *Consider the simple graph $G = (V,E)$, where $V = \{1,2,3,4,5\}$ and $E = \{12,23,34,45,15,13\}$. Hence, the associated polytope and its vertices are (we used* LattE *[11, 12] and* CDD *[14] for computational experimentation):*

$\mathcal{P}(G) = \{(r,s,t,u,v) \in \mathbb{R}^5 | r,s,t,u,v \geq 0, r+s, s+t, t+u, u+v, v+r, r+t \leq 1\}$.

$V^* = \{(0,0,0,0,0),(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,1,0),$
$(0,0,0,0,1),(1,0,0,1,0),(0,1,0,1,0),(0,1,0,0,1),(0,0,1,0,1),$
$(1/2,1/2,1/2,0,0),(1/2,1/2,1/2,1/2,0),(1/2,1/2,1/2,0,1/2),$
$(1/2,1/2,1/2,1/2,1/2)\}$,

$dim(\mathcal{P}(G)) = 5$,

$WG(n) = \frac{121}{128} + \frac{535}{192}n + \frac{219}{64}n^2 + \frac{13}{6}n^3 + \frac{45}{64}n^4 + \frac{3}{32}n^5 + (-1)^n(\frac{7}{128} + \frac{3}{64}n + \frac{1}{64}n^2)$,

$\rho(G) = \frac{1+7x+22x^2+30x^3+22x^4+7x^5+x^6}{(1-x)^3(1-x^2)^3}$,

$D(x) = \prod_{\beta \in V^*}(1 - x^{den\beta}) = (1-x)^{10}(1-x^2)^4.$

C_3 in Example 3.8 has a cycle of odd length (3) and C_4 in Example 3.9 has a cycle of even length (4). However, the graph G in Example 3.10 has three cycles of length 3, 4 and 5, two of them odd.

Remark 3.11. *If a simple graph $G = (V, E)$ with $|V| = m$ is bipartite and $\rho(G) = \frac{P(x)}{(1-x)^{m+1}}$, then coefficient vector $h(G) = (h_0, h_1, \cdots, h_m)$ of a symmetric polynomial $P(x) = h_0 + h_1x + \cdots + h_mx^m$ satisfies the following properties(see Stanley [22]):*
(1) $h_0 = 1$.
(2) $h_m = (-1)^m WG(-1)$.
(3) $min\{j \geq 0 | WG(-1) = WG(-2) = \cdots = WG(-(m-j)) = 0\}$.
$= max\{i | h_i \neq 0\}$
For example, $WC_4(-1) = WC_4(-2) = 0$ and $h(C_4) = (1, 2, 1, 0, 0)$.
(4) $WG(-n-k) = (-1)^m WG(n)$ for all n if and only if $h_i = h_{m+1-k-i}$ for all i and $h_{m+2-k-i} = h_{m+3-k-i} = \cdots = h_m = 0$.
(5) $h_i \geq 0$ for all i (Nonnegativity) (See Stanley [20, 22].)
(6) If G_1 is a subgraph of G_2, then $\mathcal{P}(G_1) \subset \mathcal{P}(G_2)$
and $h_i(G_1) \leq h_i(G_2)$. (Monotonicity)(See Stanley [20, 22].)
(7) $volume(\mathcal{P}(G)) = \frac{P(1)}{m!}$ = leading coefficient of $WG(n)$.
(See Stanley [13, 23].)
(8) All real roots α of $WG(n) = 0$ satisfy $-m \leq \alpha < \lfloor \frac{m}{2} \rfloor$.(See Beck et al [4].)

Remark 3.12. *There seem to be some connections to* **semi magic cubes**. *A semi magic cube is an $k \times k \times \cdots \times k$ table with nonnegative integral entries such that each directional sum must be equal to n (a* **magic sum**). *(The reader can find more details on semi magic squares and semi magic cubes in [5]). For example, suppose we have a 2×2 magic square. Then, the defining polytope for a 2×2 magic square with a magic sum n is a face of the polytope defining a weighted cycle graph $WC_4(n)$. In general suppose we have a 2^d magic square. Then the defining polytope for a 2^d magic square with a magic sum n is a face of the polytope defining a weighted $d-$dimensional Hypercubic Graph HC_d.*

Remark 3.13. *We can get $\rho(G)$ from the information of the graph $G = (V, E)$ using the package* **LattE** *by J.A.De Loera et. al. ([11]), and the Elliott Maple package by G. Xin ([24]), which improved the Omega package by G. Andrews et. al. ([1] or references therein(in fact, their serial articles)). We also can find the coordinates of all vertices of the polytope $P(G)$ using* cdd *and* cdd+ *by K. Fukuda ([14]).*

4 Further Questions

1. It is obvious that, for a given number m of vertices,

$$WK_m(n) \leq WG(n) \leq WD_m(n)$$

for any simple graph $G = (V, E)$ with $|V| = m$, where K_n(resp. D_m) is a complete(resp. discrete) graph of order m. We can ask the same

question for trees. That is, given the number of vertices what kind of trees T achieve the maximal or minimal values of $WT(n)$.

2. If a simple graph G is **connected** and $\rho(G) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials of lowest degree, then is $P(x)$ always symmetric ? That is, if $k = degree(P(x))$ then does $x^k P(\frac{1}{x}) = P(x)$ hold?

3. For a given simple graph G, can we express the volume of the polytope $\mathcal{P}(G)$ in terms of the graph G?

4. If $\bullet$ is an operation between two simple graphs G_1, G_2 and it is closed in the simple graphs, what is $W(G_1 \bullet G_2)(n)$ and $\rho(G_1 \bullet G_2)$? Can we get any relation between $WG(n)$ (resp. $\rho(G)$) and $W\bar{G}(n)$ (resp. $\rho(\bar{G})$)? ($\bar{G}$ is a complement graph of G).

5. Given two simple connected graphs $G_1 = (V, E_1)$ and $G_2 = (W, E_2)$, choose a vertex $v \in V$,$w \in W$. Make v and w adjacent by adding an edge between them so that two separate graphs G_1 and G_2 becomes one connected graph G. Which vertices in each side do we have to choose in order to maximize or minimize $WG(n)$?

6. Generalization or Extension of the Weighted Graph by inserting one or several slack variables in the edge between two adjacent vertices(cf. Gear Graph), or to the Problems related to (or symmetric) Magic Squares.
   i —— $\bullet$ —— $\cdots$ —— $\bullet$ —— j $\quad\Rightarrow$

   $M = (f_{ij}(n,k)) : (n+1) \times (n+1)$, where
   $$f_{ij}(n,k) = \frac{\lfloor 2^{n-i-j} \rfloor}{2^{n-i-j}} C(n+k-i-j, k)$$
   for $0 \leq i, j \leq n$. Note that $M = (f_{ij}(n,0)) = B(n)$.

7. For what kinds of rational function $f(x)$ does there exist a simple graph G so that $\rho(G) = f(x)$? In other words, what is the image $\rho(\mathcal{G})$ of the map ρ in $\mathbf{Z}[[x]]$?

8. Compute $WG(n)$ and $\rho(G)$ for other shape of simple graphs, like Wheel Graph, Cayley Graph, Complete k-partite Graph, Web Graph or Grid Graph, Regular Graphs, other Platonic Graphs etc...

References

[1] G. Andrews, P. Paule and A. Riese, *MacMahon's Partition Analysis: The Omega Package*, European J. of Combinatorics 22, 887-904, 2001.

[2] D. Avis, P. Erdös and J. Pach, Repeated Distances in the Space, *Graphs and Combinatorics* 4, 207-217, 1988.

[3] A. Barvinok, *A Course in Convexity*, Graduate Studies in Mathematics, Vol.54, AMS, 2002.

[4] M. Beck, J.A.De Loera, M. Develin, J. Pfeifle and R. Stanley, *Coefficients and Roots of Ehrhart Polynomials*, arXiv:math.CO/0402148 v1 9, Feb 2004.

[5] M. Beck and S. Robins, *Computing the Continuous Discretely Integer-Point Enumeration in Polyhedra*, Springer Undergraduate Texts in Mathematics series, to be appear.

[6] M. Bona, *Combinatorics of Permutations*, Chapman & Hall/CRC, 2004.

[7] M. Bona, *Introduction to Enumerative Combinatorics*, McGraw-Hill, 2005.

[8] M. Bona and H.-K. Ju, *Enumerating Solutions of a System of Linear Inequalities Related to Magic Squares*, to appear in Annals of Combinatorics, 2006.

[9] A. Brøndsted, *An Introduction to Convex Polytopes*, Graduate Text in Mathematics, Vol.90, Springer-Verlag, 1983.

[10] G. Csizmadia, Farthest Neighbors in Space, *Discrete Mathematics* vol.150, 81-88, 1996.

[11] J.A. De Loera, D. Haws, R. Hemmecke, P. Huggins, J. Tauzer, and R. Yoshida, *A User's guide for* latte, software package LattE and manual available at `http://www.math.ucdavis.edu/~latte/`, 2003.

[12] J.A. De Loera, R. Hemmecke, J. Tauzer, and R. Yoshida *Effective lattice point counting in rational convex polytopes*. Journal of Symbolic Computation, **38**, (2004), no 4., p 1273–1302.

[13] E. Ehrhart, *Sur les polyédres rationnels homothétiques á n dimensions*, C. R. Acad. Sci. Paris **254** (1962), 616-618.

[14] K. Fukuda, *cdd and cdd+, The CDD and CDD Plus*, available via `http://www.cs.mcgill.ca/~fukuda/soft/cdd_home/cdd.html`, 2005.

[15] R. Graham, D. Knuth and O. Pataschnik *Concrete Mathematics: A Foundation of Computer Science*, Second edition, Addison-Wesley, 1994.

[16] H.-K. Ju, *Enumeration of Weighted Complete Graphs*, submitted, 2006.

[17] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Text in Mathematics, Vol.227, Springer, 2005

[18] J. Pach and P. Agarwal, *Combinatorial Geometry*, John Wiley & Sons, Inc., 1995.

[19] A. Schrijver, *Theory of Linear and Integer Programming.* Wiley-Interscience, 1986.

[20] R. Stanley, *Combinatorics and Commutative Algebra*, 2nd edition, Progress in Mathematics, Vol. 41, Birhäuser Boston, 1996.

[21] R. Stanley, *Enumerative Combinatorics*, Vol.I., Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, 1997.

[22] R. Stanley, *A Survey of Lattice Points in Polytopes*, available at `http://www-math.mit.edu/~rstan/trans.html`, 2004.

[23] R. Stanley, *Volumes and Ehrhart Polynomials of Convex Polytopes*, available at `http://www-math.mit.edu/~rstan/trans.html`, 1999.

[24] G. Xin, *A Fast Algorithm for MacMahon's Partition Analysis*, The Electronic J. of Combinatorics 11, #R58, 2004.